A Response to Geometric Unity

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Abstract

We distill some of the technical ingredients behind "Geometric Unity", a recently proposed theory of everything by Eric Weinstein. In doing so, we identify specific gaps, both mathematical and physical in origin, which jeopardize Geometric Unity as a well-defined theory, much less one that is a candidate for a theory of everything.

1 Introduction

The crowning achievements of 20^{th} century physics have been the discovery of Quantum Field Theory and General Relativity. Famously, there as yet exists no "theory of everything" that combines the two into a consistent framework and explains the Standard Model. While a few candidates such as String Theory, M-Theory, or Loop Quantum Gravity have made various forms of progress, none have been decisive. Recently, mathematician and podcaster Eric Weinstein has put forth a video [14] expositing his theory of everything dubbed "Geometric Unity", which is a combination of his Oxford lecture of the subject dating back from 2013 and a followup presentation from 2020. Though Weinstein asserts that the theory is only partially presented, we feel that substantive comments can be made on the provided material.

We begin by reviewing Geometric Unity (henceforth GU) as presented in the video [14], which at the time of this writing, appears to be the only publicly available material on GU from its author. After this summary, we present our main concerns regarding the approach. The theory is not quantum which adds a wrinkle to its status as a theory of everything, but for the purposes of this note, we focus on the technical mathematical constructions and treat the resulting ideas as a pre-quantum classical theory. To summarize our concerns:

- GU introduces a "shiab" operator that overlooks a required complexification step. Omitting this step creates a mathematical error but including it precludes having a physically sensible quantum theory; see Section 3.1.

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- The choice of gauge group for GU naively leads to a quantum gauge anomaly, thereby rendering the quantum theory inconsistent. Any straightforward attempt to eliminate this anomaly would make the shiab operator impossible to define, compounding the previous objection; see Section 3.2.
- The setup of GU asserts that it will have supersymmetry. In 14 dimensions, adopting supersymmetry is highly restrictive. It implies that the proposed gauge group of GU cannot be correct and that the theory as stated is incomplete; see Section 3.3.
- Essential technical details of GU are omitted, leaving many of the central claims unverifiable; see Section 3.4.

The most difficult problems in mathematics and physics have always been met with serious and nonserious proposed solutions. To the extent that the two situations can be clearly delineated, those in the former group are eventually evaluated by the scientific community while those in the latter are summarily dismissed with little controversy or attention. Weinstein's situation is exceptional in that he challenges the very terms of scientific discourse [13, 4] and credit assignment [14][1] [6][2] [13][3], which itself has attracted a large popular following. It is this state of affairs that motivated our writing of this article, which we hope will enable a legitimate assessment of the work of Weinstein.

Note to the reader: In order to reference the material of GU systematically, we use footnotes to provide clickable timestamps to the video references which are aggregated at the end of the article. Since Weinstein's notation is not always consistent, our notation may sometimes deviate from that of any given timestamp.

2 The Setup

We provide definitions and constructions of the basic elements of GU, following [14].

2.1 The Spaces

Let X be a 4-dimensional manifold and let TX and T^*X denote its tangent and cotangent bundle, respectively. We have the space $U = \mathrm{Met}(X)$ consisting of all pairs $(g_{ij}(x), x)$ where $x \in X$ and $g_{ij}(x)$ is a metric on T_xX (i.e. a symmetric, positive-definite pairing on T_xX). Thus, U is a fiber bundle over X whose total space is a noncompact 14-dimensional manifold and whose sections are precisely the Riemannian metrics on X. Write $\pi : U \to X$ for the base projection map. Weinstein describes U as the "observerse".

In General Relativity, a metric has to be chosen for the underlying spacetime. For the present Euclidean setting, this requires choosing a Riemannian metric for X. If we instead consider choosing a Riemannian metric on the larger space

U, the following lemma shows that this can be achieved by choosing a connection on U.

Lemma 2.1. *Given a connection on U, regarded as a fiber bundle over X, we can assign a canonical Riemannian metric to U.*

PROOF. A point $u \in U$, can be written as $(g_{ij}(x), x)$, where $x = \pi(u)$ and $g_{ij}(x)$ is a metric at x. We have the vertical tangent space $T_u^v U$, the subspace of $T_u U$ consisting of variations along $\mathrm{Sym}_x^2(T^*X)$ on which we can define a metric using the given $g_{ij}(x)$. If we have a projection $\theta : T_u U \to T_u^v U$, its kernel, a "horizontal" subspace which we denote by $T_u^h U$, maps isomorphically to $T_x X$ via π_*. The metric on $T_x X$ then pulls back to a metric on $T_u^h U$, and we obtain a Riemannian metric on $T_u U$ from the metric on the two complementary factors $T_u^v U$ and $T_u^h U$ (we treat these factors as orthogonal). A connection on $U \to X$ is none other than a choice of horizontal subspace at every u. (Typically, one requires equivariance properties along the fibers when the fiber bundle is a principal bundle [5], but we make no such requirement here). Thus, the preceding construction shows how a connection on $U \to X$ yields a Riemannian structure on U. $\square$

Though Lemma 2.1 is verbally established in [14][4], it is unclear what its mathematical significance is. It appears Weinstein finds the lemma aesthetically significant[5]: in General Relativity one obtains a (Levi-Civita) connection from a metric while in GU, using the lemma, one obtains a metric from a connection albeit on the higher-dimensional observerse U. However, it appears Weinstein also uses Lemma 2.1 to make an invalid claim about spinors in passing[a]. This point is not relevant in what follows.

2.2 Field Content and Gauge Groups

From a Riemannian metric on U formed out of an auxiliary connection on $U \to X$ as given by Lemma 2.1, we can form a (total) spinor bundle[b] which is a 128-dimensional complex vector bundle over U. Let P be the principal $Spin(14)$-bundle associated to the vector bundle $S(U)$. Using the inclusion $Spin(14) \hookrightarrow U(128)$, we regard P as a $U(128)$-bundle.

[a] Weinstein defines the "chimeric" bundle[6] $C = T^vU \oplus \pi^*(T^*X)$, a 14-dimensional bundle over U which has a natural choice of metric: the fiber of a point $u = (g_{ij}(x), x) \in U$ determines a Riemannian metric on X which in turn determines a metric on the orthogonal factors $T_u^v U$ and $\pi_u^*(T_x^* X)$. Lemma 2.1 can be rephrased as saying that a connection on U is the same as an isomorphism between TU and C that is the identity on the subbundle T^vU. Because C has a natural metric, a spinor bundle can be defined on C with respect to that metric. Weinstein then notes that spinor bundles can be obtained on TU from this intrinsic spinor bundle on C by using the isomorphism between TU and C as parameterized by a choice of connection[7]. However, for the remainder of the talk [14], spinors are defined on the 14-dimensional space U, not the 28-dimensional spaces TU or C. In particular, the construction of spinors in GU still requires a choice of a metric or a connection on U, contrary to what Weinstein claims[8].

[b] This requires that U admits a spin structure, which is equivalent to the second Stiefel-Whitney class of TU vanishing [3].

Let $\mathcal{H}$ be the group of gauge transformations[c] of $S(U)$, i.e. the group of unitary automorphisms of $S(U)$. One can write $\mathcal{H}$ as[9]

$$\mathcal{H} = \Gamma(P \times_{Ad} U(128)) \tag{2.1}$$

where Γ denotes the space of smooth sections and the fiber product $\times_{Ad}$ denotes transition maps arising from the adjoint action

$$\mathrm{Ad}_g(h) = ghg^{-1}.$$

Recall that gauge transformations form the group of local symmetries with which all physically meaningful expressions must be invariant. In GU, (2.1) is regarded as the group of homogeneous gauge transformations sitting inside a larger group defined as follows. Let $\mathcal{A}$ be the space of connections[d] on $S(U)$. The bundle $S(U)$ has a canonical spin connection A_0 induced from the Levi-Civita connection on TU. Thus, we can identify $\mathcal{A}$ with $\Omega^1(\mathrm{Ad}(P))$ (where $\mathrm{Ad}(P)$ is the adjoint bundle associated to P with fiber equal to $\mathfrak{u}(128)$) via the map

$$A \mapsto \pi := A - A_0.$$

Henceforth, we can identify connections A with their associated 1-forms π.

Define the *group of inhomogeneous gauge transformations* via the semi-direct product[10]

$$\mathcal{G} = \mathcal{H} \ltimes \Omega^1(\mathrm{Ad}(P))$$

where multiplication is given by

$$(h_1, \pi_1) \cdot (h_2, \pi_2) = (h_1 h_2, \mathrm{Ad}_{h_2^{-1}}(\pi_1) + \pi_2).$$

Define an embedding of $\mathcal{H}$ into $\mathcal{G}$ via

$$\begin{aligned} \tau = \tau_{A_0} : \mathcal{H} &\to \mathcal{G} \\ h &\mapsto (h, h^{-1} d_{A_0} h). \end{aligned}$$

The image of τ is called $\mathcal{H}_\tau$, the *tilted group of gauge transformations*[11]. Note that another way to interpret $h^{-1} d_{A_0} h$ is as the pullback via $h : U \to P \times_{Ad} U(128)$ of the induced connection 1-form A_0 on $P \times_{Ad} U(128)$. This interpretation makes it manifest that τ is a homomorphism.

A key aspect of GU is that all objects are invariant with respect to the action of $\mathcal{H}$, either via its natural adjoint action on $\Omega^\bullet(\mathrm{Ad}(P))$ or else via the action given by composition of τ and the adjoint action of $\mathcal{G}$ on itself.

[c] For linguistic clarity, we use the term gauge group to denote the (finite-dimensional) structure group of a principal bundle and the term group of gauge transformations to denote the (infinite-dimensional) group of a local symmetries of a bundle.

[d] From the double cover $Spin(n) \to SO(n)$, connections on TU are in one-to-one correspondence with spin connections on $S(U)$, which form a subspace within the space $\mathcal{A}$.

2.3 The Equations of Motion

Having defined the basic ingredients of GU, we now write down the equations of motion. First, given a connection $A \in \mathcal{A}$, we have its curvature tensor

$$F_A = dA + A \wedge A.$$

Replicating an abuse of notation in the GU video, we also write F_π to denote $F_{A_0+\pi}$. Next, we define the *augmented torsion tensor*[e]:

$$\begin{aligned} T : \mathcal{G} &\to \Omega^1(\mathrm{Ad}(P)) \\ (\epsilon, \pi) &\mapsto \pi - \epsilon^{-1} d_{A_0} \epsilon. \end{aligned}$$

Next we "define" a family of zeroth order operators called *shiab operators*[f]. Fix $\Phi_i \in \Omega^i(\mathrm{Ad}(P))$. Then we can define[12]

$$\begin{aligned} \odot_\epsilon : \Omega^k(\mathrm{Ad}(P)) &\to \Omega^{k+i}(\mathrm{Ad}(P)) \\ \eta &\mapsto [\mathrm{Ad}_{\epsilon^{-1}}(\Phi_i) \wedge \eta]. \end{aligned}$$

The equations of motion for GU are given by[13]

$$\odot_\epsilon F_\pi + [\odot_\epsilon T, T] + *T = 0,$$

where the Shiab operator makes use of a $d-3$-form and $*$ is the Hodge star operator. The left-hand side is a gauge-equivariant map from $\mathcal{G}$ to to $\Omega^{d-1}(\mathrm{Ad}(P))$. This follows from the gauge-equivariance of the curvature tensor and the gauge-equivariance of the augmented torsion tensor[14].

3 Objections to Geometry Unity

With the relevant objects and notation of GU defined in the previous section, we now focus on the shortcomings of GU.

3.1 Shiab Operators and Complexification

In order to define the shiab operator, Weinstein needs to choose a field[15] Φ which he defines vaguely as follows[16]. Choose $\Phi_i \in \Omega^i(\mathrm{Ad}(P))$ that is "pure trace" arising from the fact that we have "the same representation" arising from the "auxiliary directions" (the fibers of $\mathrm{Ad}(P)$) and the "geometric directions" (the base space U). We understand this description to mean the following:

First, note that the fibers of both $\mathrm{Ad}(P)$ and $\Lambda^\bullet(T^*U)$ (the exterior algebra on T^*U) are 2^{14}-dimensional; the first due to the fibers of $\mathrm{Ad}(P)$ being $\mathfrak{u}(128)$, the second due to the fact that U is 14-dimensional. From this, we have the natural isomorphism of bundles

$$\mathrm{Ad}(P) \otimes \mathbb{C} \cong \Lambda^\bullet(T^*U) \otimes \mathbb{C}, \tag{3.1}$$

[e] Note that π is the contorsion tensor of Riemannian geometry.
[f] We will see where this putative definition falls apart in the next section.

where we have performed a complexification of our bundles as indicated via the notation "$\otimes\,\mathbb{C}$". Indeed, both bundles of (3.1) are isomorphic to $V :=$ $\mathrm{End}(S(U))$, the bundle of complex endomorphisms of the spinor bundle (it is a bundle whose fiber is isomorphic to the algebra of 128×128 complex matrices). This leads to a natural isomorphism of representations of $\mathcal{H}$:

$$\Gamma(\mathrm{Ad}(P) \otimes \mathbb{C}) \cong \Gamma(\Lambda^\bullet(T^*U) \otimes \mathbb{C}).$$

At the level of fibers, the representation is given via the conjugation of complex matrices by a unitary transformation. Hence, by Φ being "pure trace" according to Weinstein, we interpret the statement to mean that Φ is meant to be a section of the trivial representation inside $V \otimes V^*$ given by the scalar transformations of V.

However, note that we cannot define the shiab operator without complexification since in general we have

$$\mathrm{Ad}(P) \ncong \Lambda^\bullet(T^*U).$$

This follows from the fact that there is no natural vector space isomorphism between $\mathfrak{u}(128)$ and the real Clifford algebra in dimension 14 (which is the algebra of 128×128 real matrices [3]). Thus, we are left with the conclusion that Weinstein has either failed to mention or annotate a complexification step or else has made a fundamental error. If we are charitable and assume the former, then this immediately raises problems as a pre-quantum classical theory.

Indeed, following [15], by complexifying the space of connections (and hence the gauge group), the resulting quantum field theory will either fail to be unitary (quantum operators will not be Hermitian) or else result in a Hamiltonian that has energy spectrum unbounded in both the positive and negative directions. Neither option is tenable.

3.2 Chiral Anomaly in the GU Gauge Group

As stated in subsection 2.2, the gauge group of GU is the unitary transformations acting on spinors in 14 dimensions, i.e. $U(128)$, which is then further refined into the "tilted gauge group". However, there is an issue with gauging this particular group, due to the chiral anomaly present in even dimensions [8]. Namely, since the entire group of unitary transformations serves as the gauge group (rather than than the usual spin group), a subgroup of the gauge group consists of axial transformations $\psi \to \exp(\bar{\gamma}\theta)\psi$, where $\bar{\gamma}$ is the chirality operator in 14 dimensions (i.e. $\bar{\gamma}\psi_{L/R} = \pm\psi_{L/R}$ for the left and right-handed Weyl spinors). Since the gauge group $U(128)$ also contains a gauge connection associated to the central $U(1)$ subgroup of $U(128)$, this gauge connection induces an abelian chiral gauge anomaly [8]. This breaks invariance under the axial transformation and so the associated connection will have a gauge anomaly. We are unaware of any possible anomaly cancellation mechanism that could remedy this. Since gauge anomalies break unitarity, this renders the quantum theory ill-defined for GU [12].

Recall that $U(128)$ arose from the choice of structure group of the bundle P. When P was introduced in Section 2.2, we remarked that it was actually a $Spin(14)$-bundle, in which case the use of $Spin(14)$ instead of $U(128)$ would circumvent the aforementioned anomaly. But using a $Spin(14)$ structure group would render (3.1) impossible for basic dimensionality reasons and the shiab operators could not be defined. Thus, it is impossible to both remove the gauge anomaly and have a well-defined shiab operator, thereby rendering GU inconsistent.

3.3 Dimensionality and Supersymmetry Constraints

Weinstein claims that GU exhibits a supersymmetric extension to the "inhomogeneous gauge group"[17]. There are, however, very strong limitations on the kinds of symmetries and interactions allowed between particles based on the dimensionality of the underlying space. Although there is a paucity of details on the exact superalgebra "extension" that is envisioned, the fact that claims are made about spin-$\frac{1}{2}$ and Rarita-Schwinger spin-$\frac{3}{2}$ fields[18] is sufficient for the following argument to apply.

We relay these important facts about higher-dimensional supersymmetry and higher spin theories:

1. Any supersymmetric theory in dimension $D \geq 12$ necessarily has a spin-3 field even in its lowest spin rep, owing purely to representation theory [7] (for modern reviews, see [1] and in particular section 8.2 of [10]). In the case of a theory with spin-1 or spin-2 fields (i.e. with dynamical gauge connections or metric tensors), this will necessitate at least a massless spin-3 field.

2. A theory with a spin-3 field necessarily contains an infinite tower of higher-spin fields due to known inconsistencies of a finite number of interacting fields with any spin greater than 2 (for a modern review and references therein, see [9, 2]). The only known higher spin theories come from String Theory and Vasiliev higher spin theory (HST), and creating HSTs typically requires carefully evading many extremely non-trivial no-go theorems [9, 11].

3. Higher spin gauge theories must have infinite-dimensional structure groups. Since the structure group of GU has been asserted to be exclusively[19] $U(128)$, this leads to an inconsistency. The massless higher spin fields each will require their own gauge symmetry in the structure group [11, 9], where at best $U(128)$ or $Spin(14)$ could be a subgroup of this infinite-dimensional structure supergroup.

Thus once the inhomogeneous gauge group has been extended with supersymmetry, there is an infinite sector of fields that has not been accounted for and the naive gauge group of transformations cannot be based purely on a $U(128)$ structure group.

3.4 Numerous Omissions

One of the most significant features of GU is that it is supposed to provide a unification of Einstein's equations[20], the Yang-Mills equations[21], and the Dirac equation[22]. Unfortunately, the details for this unification, as far as the authors can tell, are hardly provided and thus the central insights of the theory are not possible to verify. For example, Weinstein claims that certain differential forms are exact[23] and that certain sign calculations work out[24] but he does not supply the details. Weinstein also provides a list of issues that GU has towards the end of [14][25], one of which is that the theory lives on U, a 14-dimensional space, and it is unclear how to recover a theory that is applicable to our 4-dimensional universe. Our conclusion is that, even supposing the previous technical concerns could be addressed, the volume of missing or inexplicit computations renders the formulation of GU largely incomplete.

4 Conclusions

Freeman Dyson said "It is better to be wrong than to be vague." A good justification for this dictum is that truth often arises from a well-discerned error, especially when it is aided by the help of others. Every scientific theory has its flaws, but those that have stood the test of time have done so by being developed through the collective efforts of the scientific community. We hope our response is an encouragement to Weinstein to provide further clarity to his ideas, ideally as a technical paper.

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A Timestamps (from [14] unless otherwise noted)

Section 1

1. 28:33 2:46:14
2. [6] 1:24:40
3. [13] 2:01:49

Section 2

4. 1:13:10
5. 1:15:53, 2:48:22
6. 1:12:26
7. 1:13:54
8. 1:14:08
9. 1:22:40
10. 1:26:16, 2:27:06
11. 1:28:00
12. 1:35:53
13. 1:43:35
14. 2:33:20

Section 3

15. 2:33:41
16. 1:34:55
17. Stated in a non-technical manner at 1:30:19 and 2:24:50. This was directly clarified in this PBS interview at 1:06:32.
18. 2:41:15
19. 1:22:04
20. 1:44:08
21. 2:02:19
22. 1:57:47, 2:02:58
23. 1:45:40
24. 2:03:32
25. 2:40:02